Continuum Mechanics

Chapter 8

Plasticity

C. Agelet de Saracibar

ETS Ingenieros de Caminos, Canales y Puertos, Universidad Politécnica de Cataluña (UPC), Barcelona, Spain

International Center for Numerical Methods in Engineering (CIMNE), Barcelona, Spain
Chapter 8 · Plasticity

1. Introduction
2. 1D Plasticity model
3. 3D Plasticity model
4. Yield surfaces
Introduction

Plasticity may be characterized by the following main features,

- **Nonlinear** stress-strain relationship
- **Lack of unicity** in the stress-strain relationship
- Presence of *irreversible strains*, i.e. *plastic strains*, in a loading-unloading cycle

Here we will consider plasticity within an infinitesimal strains framework.
Space of Principal Stresses

Consider a Cartesian system framework defined by the three principal stresses, defining the *space of principal stresses*, such that the stress state at an arbitrary point of a continuum body, characterized by the principal stresses \( \sigma_1, \sigma_2, \sigma_3 \), may be represented by a point \( P(\sigma_1, \sigma_2, \sigma_3) \) in the space of principal stresses.
Hydrostatic Stress Axis

The *hydrostatic stress axis* is defined by the collection of points belonging to the space of principal stresses that satisfy the condition $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_m$. 

\[
\sigma_1 = \sigma_2 = \sigma_3 = \sigma_m
\]
Octahedral Plane

An octahedral plane is defined as a plane that is orthogonal to the hydrostatic stress axis. The equation of an octahedral plane containing the point $P$ is given by,

$$\sigma_1 + \sigma_2 + \sigma_3 = \text{constant}$$
Unit Normal to the Octahedral Plane

The unit normal vector to the octahedral plane, denoted as $\mathbf{n}$, is given by,

$$[\mathbf{n}] = \frac{1}{\sqrt{3}} [1, 1, 1]^T$$
**Introduction**

Space of Principal Stressess

\[ \sigma_1 = \sigma_2 = \sigma_3 = \sigma_m \]

\[ A(\sigma_m, \sigma_m, \sigma_m) \]

\[ P(\sigma_1, \sigma_2, \sigma_3) \]
Octahedral Normal Stress

The octahedral normal stress, denoted as $\sigma_{oct}$, is a scalar-valued quantity defined as the projection of the vector position $OP$ along the unit normal $n$,

$$\sqrt{3}\sigma_{oct} := \overrightarrow{OP} \cdot n = \|\overrightarrow{OA}\| \text{sgn} \sigma_m$$
Octahedral Normal Stress

\[
\sqrt{3\sigma_{oct}} := \overrightarrow{OP} \cdot \mathbf{n} = \left| \overrightarrow{OA} \right| \text{sgn} \sigma_m = \sqrt{3} \sigma_m
\]

\[
\sigma_{oct} := \sigma_m = \frac{1}{3} I_1 = \frac{1}{3} \text{tr} \mathbf{\sigma}
\]
Octahedral Normal Stress Surface

The *domain in the space of principal stresses* satisfying the condition $I_1 = \text{constant}$, is the *octahedral plane* at a distance $d$ of the origin given by,

$$d = \sqrt{3} \sigma_{oct} = \sqrt{3} \sigma_m = \frac{\sqrt{3}}{3} I_1$$
Pure Deviatomic Stress States

*Pure deviatomic stress states*, satisfying $\sigma_{oct} = 0$, are characterized in the space of principal stresses by points located on the *octahedral plane which contains the origin*, satisfying the condition $I_1 = 0$,.

$$\sigma_{oct} = 0 \quad \Rightarrow \quad d = \sqrt{3} \sigma_{oct} = \sqrt{3} \sigma_m = \frac{\sqrt{3}}{3} I_1 = 0$$
Octahedral Shear Stress

The *octahedral shear stress*, denoted as $\tau_{oct}$, is a positive scalar-valued quantity defined as,

$$\sqrt{3\tau_{oct}} := |\overrightarrow{AP}|$$
Octahedral Shear Stress

\[
\left( \sqrt{3} \sigma_{oct} \right)^2 = \| \overrightarrow{AP} \|^2 = \| \overrightarrow{OP} \|^2 - \| \overrightarrow{OA} \|^2 \\
= \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \frac{1}{3} \left( \sigma_1 + \sigma_2 + \sigma_3 \right)^2 \\
= 2J_2'
\]

\[
\sigma_{oct} = \sqrt{\frac{2}{3}} \left( J_2' \right)^{1/2}
\]
Octahedral Shear Stress Surface

The domain in the space of principal stresses satisfying the condition $J'_2 = \text{constant}$, is the cylinder with axis the hydrostatic stress axis and radius given by

$$R = \sqrt{3} \tau_{oct} = \sqrt{2} \left( J'_2 \right)^{1/2}$$
Spherical Stress States

*Spherical stress states*, satisfying $\tau_{oct} = 0$, are characterized in the space of principal stresses by points located along the *hydros-tatic stress axis*, satisfying the condition $J'_2 = 0$,

$$\tau_{oct} = 0 \quad \Rightarrow \quad R = \sqrt{3} \tau_{oct} = \sqrt{2} \left( J'_2 \right)^{1/2} = 0$$
Principal Stress Space

A stress state at a point of a continuum medium may be characterized in the principal stress space by a point with coordinates the principal stresses.

Alternatively, the point in the principal stress space may be characterized by the stress invariants $I_1, J'_2, J'_3$:

- The first invariant of the stress tensor $I_1$ fixes a given octahedral plane at a distance $d = \sqrt{3}I_1/3$ of the origin.
- The second invariant of the deviatoric stress tensor $J'_2$ fixes a circumference on the octahedral plane with radius $R = \sqrt{2(J'_2)^{1/2}}$ and center on the point of the hydrostatic stress axis.
- The third invariant of the deviatoric stress tensor $J'_3$ fixes a point of the circumference.
**Assignment 8.1**

Determine the shape of the surface which in the principal stress space is given by,

\[ aI_1^2 + bJ'_2 = c \quad a, b, c > 0 \]
Assignment 8.2

The shape of a surface in the principal stress space is an axisymmetric ellipsoid along the hydrostatic stress axis. The center of the ellipsoid is at the origin. Semi-axes along the hydrostatic stress axis and on the deviatoric plane (octahedral plane passing through the origin) are $a$ and $b$, respectively. Obtain the equation of the surface in terms of the appropriate stress invariants.
1D Rate-independent Perfect Plasticity Model

H1. Additive split of the infinitesimal strain

The infinitesimal strain may be additively split into an elastic and a plastic part,

\[ \varepsilon = \varepsilon^e + \varepsilon^p \]

Similarly, the infinitesimal strain rate can be additively split into an elastic strain rate and a plastic strain rate part,

\[ \dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p \]
1D Rate-independent Perfect Plasticity Model

H2. Linear elastic response

The elastic strain satisfies the *linear elastic constitutive equation*,

\[
\varepsilon^e = \frac{1}{E} \sigma, \quad \sigma = E \varepsilon^e = E (\varepsilon - \varepsilon^p)
\]

where \( E \) is the *elastic Young’s modulus*.

Similarly the elastic strain rate satisfies the linear elastic rate constitutive equation,

\[
\dot{\varepsilon}^e = \frac{1}{E} \dot{\sigma}, \quad \dot{\sigma} = E \dot{\varepsilon}^e = E (\dot{\varepsilon} - \dot{\varepsilon}^p)
\]
H3. Space of admissible stresses, elastic domain and yield surface

The space of admissible stresses, elastic domain and yield surface are defined, respectively, as,

\[
\mathcal{E}_\sigma := \left\{ \sigma \in \mathbb{R} \mid f(\sigma) \leq 0 \right\}
\]

\[
\text{int}\left( \mathcal{E}_\sigma \right) := \left\{ \sigma \in \mathbb{R} \mid f(\sigma) < 0 \right\}
\]

\[
\partial \mathcal{E}_\sigma := \left\{ \sigma \in \mathbb{R} \mid f(\sigma) = 0 \right\}
\]

where the yield function, denoted as \( f(\sigma) \), is given by,

\[
f(\sigma) := |\sigma| - \sigma_Y \leq 0
\]

and \( \sigma_Y > 0 \) is the yield stress of the material.
H4. Plastic flow rule

Plastic flow rule provides the evolution equation for the plastic strain such that,

\[
\begin{align*}
\text{if } \sigma &= \sigma_y > 0 \quad \Rightarrow \quad \dot{\varepsilon}^p = \gamma \geq 0 \\
\text{if } \sigma &= -\sigma_y < 0 \quad \Rightarrow \quad \dot{\varepsilon}^p = -\gamma \leq 0
\end{align*}
\]

where the plastic multiplier or plastic consistency parameter, denoted as \( \gamma \geq 0 \), is a non-negative scalar valued quantity. *Plastic flow rule* can be cast into a single expression yielding,

\[
\text{if } f(\sigma) = 0 \quad \Rightarrow \quad \dot{\varepsilon}^p = \gamma \text{sgn}(\sigma) = \gamma \partial_\sigma f(\sigma)
\]
H5. Kuhn-Tucker loading/unloading conditions

Plastic loading and elastic loading/unloading may be characterized by the following conditions,

\[
\text{if } \gamma > 0 \quad \Rightarrow \quad f(\sigma) = 0 \\
\text{if } f(\sigma) < 0 \quad \Rightarrow \quad \gamma = 0
\]

The above conditions may be cast into the so called Kuhn-Tucker loading/unloading conditions, yielding,

\[
\gamma \geq 0, \quad f(\sigma) \leq 0, \quad \gamma f(\sigma) = 0
\]
H6. Plastic consistency condition

Plastic consistency may be characterized by the following conditions,

\[
\text{if } f(\sigma) = 0 \quad \text{and} \quad \gamma > 0 \quad \Rightarrow \quad \dot{f}(\sigma) = 0
\]

\[
\text{if } f(\sigma) = 0 \quad \text{and} \quad \dot{f}(\sigma) < 0 \quad \Rightarrow \quad \gamma = 0
\]

The above conditions may be cast into the so called plastic consistency conditions, yielding,

\[
\text{if } f(\sigma) = 0 \quad \Rightarrow \quad \gamma \geq 0, \quad \dot{f}(\sigma) \leq 0, \quad \gamma \dot{f}(\sigma) = 0
\]
Plastic consistency parameter

The non-trivial value of the plastic multiplier or plastic consistency parameter can be obtained imposing the plastic consistency condition for the non-trivial case of plastic loading, yielding,

\[
\text{if } f(\sigma) = 0 \text{ and } \gamma > 0 \Rightarrow \dot{f}(\sigma) = 0
\]

Taking the time derivative of the yield function yields,

\[
\dot{f}(\sigma) = \partial_\sigma f(\sigma) \dot{\sigma} = \text{sign}(\sigma) \dot{\sigma} = \text{sign}(\sigma) E \left( \dot{\varepsilon} - \dot{\varepsilon}^p \right)
\]

\[
= \text{sign}(\sigma) E \dot{\varepsilon} - \text{sign}(\sigma) E \text{sign}(\sigma) \gamma = \text{sign}(\sigma) E \dot{\varepsilon} - E \gamma
\]

\[
= \text{sign}(\sigma) \dot{\sigma}^{\text{trial}} - E \gamma = 0
\]

\[
\gamma = E^{-1} \text{sign}(\sigma) \dot{\sigma}^{\text{trial}} = E^{-1} \partial f(\sigma) \dot{\sigma}^{\text{trial}}, \quad \dot{\sigma}^{\text{trial}} := E \dot{\varepsilon}
\]
Plastic consistency parameter

*Plastic loading/elastic unloading* from the yield surface can be determined in terms of the projection of the trial stress rate along the unit normal to the yield surface,

Elastic unloading:

\[
\partial_\sigma f(\sigma) \dot{\varepsilon}_{\text{trial}} < 0 \quad \Rightarrow \quad \begin{cases} 
\gamma = 0 \\
\dot{f}(\sigma) = \partial_\sigma f(\sigma) \dot{\varepsilon}_{\text{trial}} < 0 \\
\gamma \dot{f}(\sigma) = 0
\end{cases}
\]

Plastic loading:

\[
\partial_\sigma f(\sigma) \dot{\varepsilon}_{\text{trial}} > 0 \quad \Rightarrow \quad \begin{cases} 
\gamma = E^{-1} \partial_\sigma f(\sigma) \dot{\varepsilon}_{\text{trial}} > 0 \\
\dot{f}(\sigma) = \partial_\sigma f(\sigma) \dot{\varepsilon}_{\text{trial}} - E \gamma = 0 \\
\gamma \dot{f}(\sigma) = 0
\end{cases}
\]
Plastic consistency parameter

The *plastic multiplier* or *plastic consistency parameter* can be cast into a single expression, covering both the (non-trivial) plastic loading and the (trivial) elastic unloading cases, yielding,

\[
\gamma = E^{-1} \langle \partial_\sigma f(\sigma) \dot{\sigma}^{\text{trial}} \rangle \geq 0
\]
1D Rate-independent Hardening Plasticity Model

H1. Additive split of the infinitesimal strain

The infinitesimal strain may be additively split into an elastic and a plastic part,

\[ \varepsilon = \varepsilon^e + \varepsilon^p \]

\[ \dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p \]

H2. Isotropic and kinematic hardening variables

Isotropic and kinematic hardening variables in the strain space are denoted, respectively as \( \xi, \xi \). Their conjugate variables in the stress space are denoted, respectively, as \( q, \overline{q} \).
**1D Plasticity Model**

**H3. Linear elastic and linear hardening response**

The *elastic strain* satisfies the *linear elastic constitutive equation*,

\[
\varepsilon^e = \frac{1}{E} \sigma, \quad \sigma = E \varepsilon^e = E (\varepsilon - \varepsilon^p)
\]

\[
\dot{\varepsilon}^e = \frac{1}{E} \dot{\sigma}, \quad \dot{\sigma} = E \dot{\varepsilon}^e = E (\dot{\varepsilon} - \dot{\varepsilon}^p)
\]

where \( E \) is the *elastic Young’s modulus*.

The *isotropic/kinematic hardening* variables satisfy the following *linear constitutive hardening equations*,

\[
q = -H \dot{\xi}, \quad \bar{q} = -K \bar{\dot{\xi}}
\]

where \( H \) and \( K \) are the constant *isotropic* and *kinematic hardening parameters*, respectively.
H5. Associative plastic flow rule

Associative plastic flow rule equations may be written as,

\[
\begin{align*}
\dot{\varepsilon}^p &= \gamma \partial_\sigma f(\sigma, q, \bar{q}) = \gamma \text{sgn}(\sigma - \bar{q}) \\
\dot{\xi} &= \gamma \partial_q f(\sigma, q, \bar{q}) = \gamma \\
\ddot{\xi} &= \gamma \partial_{\bar{q}} f(\sigma, q, \bar{q}) = -\gamma \text{sgn}(\sigma - \bar{q})
\end{align*}
\]

H6. Kuhn-Tucker loading/unloading conditions

Kuhn-Tucker loading/unloading conditions may be written as,

\[
\gamma \geq 0, \quad f(\sigma, q, \bar{q}) \leq 0, \quad \gamma f(\sigma, q, \bar{q}) = 0
\]
H7. Plastic consistency condition

Plastic consistency conditions may be written as,

\[
\text{if } f(\sigma, q, \bar{q}) = 0 \quad \Rightarrow \quad \gamma \geq 0, \quad \dot{f}(\sigma, q, \bar{q}) \leq 0, \quad \gamma \dot{f}(\sigma, q, \bar{q}) = 0
\]
Plastic consistency parameter

The non-trivial value of the plastic multiplier or plastic consistency parameter can be obtained imposing the plastic consistency condition for the non-trivial case of plastic loading, yielding,

\[
\text{if } \quad f(\sigma, q, \overline{q}) = 0 \quad \text{and} \quad \gamma > 0 \quad \Rightarrow \quad \dot{f}(\sigma, q, \overline{q}) = 0
\]

Taking the time derivative of the yield function yields,

\[
\dot{f} = \partial_{\sigma} f \dot{\sigma} + \partial_{q} f \dot{q} + \partial_{\overline{q}} f \dot{\overline{q}}
\]

\[
= \partial_{\sigma} f E(\dot{\varepsilon} - \dot{\varepsilon}^p) - \partial_{q} f H \dot{\xi} - \partial_{\overline{q}} f K \dot{\overline{\xi}}
\]

\[
= \partial_{\sigma} f E\dot{\varepsilon} - \gamma \left( \partial_{\sigma} f E \partial_{\sigma} f + \partial_{q} f H \partial_{q} f + \partial_{\overline{q}} f K \partial_{\overline{q}} f \right) = 0
\]
Introducing the trial stress and computing the derivatives of the yield function with respect to the stress and hardening variables in the stress space, yields,

\[
\sigma^{trial} := E\varepsilon, \quad \dot{\sigma}^{trial} := E\dot{\varepsilon}
\]

\[
\partial_{\sigma}f = \text{sgn}(\sigma - q), \quad \partial_{q}f = 1, \quad \partial_{q}f = -\text{sgn}(\sigma - q)
\]

\[
\dot{f} = \partial_{\sigma}f \cdot \dot{\sigma}^{trial} - \gamma(E + H + K) = 0
\]

\[
\gamma = (E + H + K)^{-1} \partial_{\sigma}f \cdot \dot{\sigma}^{trial} > 0
\]
Plastic consistency parameter

*Plastic loading/elastic unloading* from the yield surface can be determined in terms of the projection of the trial stress rate along the unit normal to the yield surface,

Elastic unloading:

\[ \partial_{\sigma} f \cdot \dot{\sigma}^{trial} < 0 \quad \Rightarrow \quad \begin{cases} 
\gamma = 0 \\
\dot{f} = \partial_{\sigma} f \cdot \dot{\sigma}^{trial} < 0 \\
\gamma \dot{f} = 0
\end{cases} \]

Plastic loading:

\[ \partial_{\sigma} f \cdot \dot{\sigma}^{trial} > 0 \quad \Rightarrow \quad \begin{cases} 
\gamma = E^{-1} \partial_{\sigma} f \cdot \dot{\sigma}^{trial} > 0 \\
\dot{f} = \partial_{\sigma} f \cdot \dot{\sigma}^{trial} - E\gamma = 0 \\
\gamma \dot{f} = 0
\end{cases} \]
Plastic consistency parameter

The *plastic multiplier* or *plastic consistency parameter* can be cast into a single expression, covering both the (non-trivial) plastic loading and the (trivial) elastic unloading cases, yielding,

\[
\gamma = \left( E + H + K \right)^{-1} \langle \partial_\sigma f(\sigma, q, \bar{q}) \cdot \dot{\sigma}^{trial} \rangle \geq 0
\]
Continuum elastoplastic tangent modulus

For the (non-trivial) case of plastic loading, the stress rate and continuum elastoplastic tangent modulus are given by,

\[ \dot{\sigma} = E\dot{\epsilon}^e = E\dot{\epsilon} - E\dot{\epsilon}^p = E\dot{\epsilon} - E\gamma \partial_\sigma f \]

\[ = \dot{\sigma}^{trial} - E\left( E + H + K \right)^{-1} \partial_\sigma f \dot{\sigma}^{trial} \partial_\sigma f \]

\[ = \dot{\sigma}^{trial} - E\left( E + H + K \right)^{-1} \dot{\sigma}^{trial} = \left( 1 - E\left( E + H + K \right)^{-1} \right) \dot{\sigma}^{trial} \]

\[ = E\left( 1 - E\left( E + H + K \right)^{-1} \right) \dot{\epsilon} := E^{ep} \dot{\epsilon} \]

\[ E^{ep} = E\left( 1 - E\left( E + H + K \right)^{-1} \right) \]
Differential constitutive equation

For *elastic loading/unloading* cases the differential constitutive equation may be written as,

\[ \text{if } f < 0 \text{ or } f = 0 \text{ and } \dot{f} < 0 \implies d\sigma = E d\varepsilon \]

For the *plastic loading* case the differential constitutive equation may be written as,

\[ \text{if } f = 0 \text{ and } \dot{f} = 0 \implies d\sigma = E^{ep} d\varepsilon \]
Assignment 8.3

The bar-truss structure of the figure consists of three bars of the same length $L$. The three of them are modeled as an elastic-perfect plastic material model with elastic Young’s modulus $E$, yield stresses $\sigma_0$ in traction and $10\sigma_0$ in compression.
Assignment 8.3

An increasing vertical load $P$ is applied at the point $O$, until the vertical displacement of point $O$, denoted as $\delta$, takes the value,

$$ \delta = 20 \frac{\sigma^0}{E} l $$

Then the structure is fully unloaded, until the load is $P=0$.

1) Plot the curve $P$ vs $\delta$ for the full loading-unloading cycle. Indicate the state of the bars (elastic or plastic) during the full loading-unloading cycle.

2) Compute the remaining value of the vertical displacement of point $O$, denoted as $\delta$, at the end of the loading-unloading cycle.
3D Rate-independent Perfect Plasticity Model

H1. Additive split of the infinitesimal strain tensor
The infinitesimal strain tensor may be additively split into an elastic strain tensor and a plastic strain tensor,

\[ \varepsilon = \varepsilon^e + \varepsilon^p \]

\[ \dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p \]

H2. Linear elastic response
The linear elastic constitutive equation may be written as,

\[ \sigma = C : \varepsilon^e = C : (\varepsilon - \varepsilon^p) \]

\[ \dot{\sigma} = C : \dot{\varepsilon}^e = C : (\dot{\varepsilon} - \dot{\varepsilon}^p) \]

where \( C \) is the fourth-order constant elastic constitutive tensor.
H3. Space of admissible stresses, elastic domain and yield surface

The space of admissible stresses, elastic domain and yield surface are defined, respectively, as,

$$\mathcal{E}_\sigma := \{ \sigma \in \mathbb{S} \mid f(\sigma) \leq 0 \}$$

$$\text{int}(\mathcal{E}_\sigma) := \{ \sigma \in \mathbb{S} \mid f(\sigma) < 0 \}$$

$$\partial \mathcal{E}_\sigma := \{ \sigma \in \mathbb{S} \mid f(\sigma) = 0 \}$$

where the yield function, denoted as $f(\sigma)$, is a scalar-valued function of the stress tensor which may be viewed as a surface in the space of principal stresses.
H4. Associative plastic flow rule

*Associative plastic flow rule*, defining the evolution of the plastic strain tensor, may be written as,

\[
\dot{\varepsilon}^p = \gamma \partial_\sigma f(\sigma)
\]

where \( \gamma \geq 0 \) is the *plastic multiplier* or *plastic consistency parameter*.

H5. Kuhn-Tucker loading/unloading conditions

*Kuhn-Tucker loading/unloading conditions* may be written as,

\[
\gamma \geq 0, \quad f(\sigma) \leq 0, \quad \gamma f(\sigma) = 0
\]
H6. Plastic consistency condition

*Plastic consistency condition* may be written as,

\[ f(\mathbf{\sigma}) = 0 \quad \Rightarrow \quad \gamma \geq 0, \quad \dot{f}(\mathbf{\sigma}) \leq 0, \quad \gamma \dot{f}(\mathbf{\sigma}) = 0 \]
Plastic consistency parameter

The non-trivial value of the plastic multiplier or plastic consistency parameter can be obtained imposing the plastic consistency condition for the non-trivial case of plastic loading, yielding,

\[
\text{if } f(\sigma) = 0 \quad \text{and} \quad \gamma > 0 \quad \Rightarrow \quad \dot{f}(\sigma) = 0
\]

Taking the time derivative of the yield function yields,

\[
\dot{f} = \partial_\sigma f : \dot{\sigma} = \partial_\sigma f : C : (\dot{\varepsilon} - \dot{\varepsilon}^p) = \partial_\sigma f : C : \dot{\varepsilon} - \gamma (\partial_\sigma f : C : \partial_\sigma f)
\]

\[
= \partial_\sigma f : \dot{\sigma}^{trial} - \gamma (\partial_\sigma f : C : \partial_\sigma f) = 0
\]

The plastic multiplier or plastic consistency parameter is given by,

\[
\gamma = (\partial_\sigma f : C : \partial_\sigma f)^{-1} \partial_\sigma f : \dot{\sigma}^{trial} \geq 0
\]
Plastic consistency parameter

*Plastic loading/elastic unloading* from the yield surface can be determined in terms of the projection of the trial stress rate tensor along the unit normal to the yield surface,

Elastic unloading:
\[ \partial_{\sigma} f : \dot{\sigma}^{trial} < 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} \gamma = 0 \\ \dot{f} = \partial_{\sigma} f : \dot{\sigma}^{trial} < 0 \\ \gamma \dot{f} = 0 \end{array} \right. \]

Plastic loading:
\[ \partial_{\sigma} f : \dot{\sigma}^{trial} > 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} \gamma = (\partial_{\sigma} f : C : \partial_{\sigma} f)^{-1} \partial_{\sigma} f : \dot{\sigma}^{trial} > 0 \\ \dot{f} = \partial_{\sigma} f : \dot{\sigma}^{trial} - \gamma (\partial_{\sigma} f : C : \partial_{\sigma} f) = 0 \\ \gamma \dot{f} = 0 \end{array} \right. \]
Plastic consistency parameter

The plastic multiplier or plastic consistency parameter can be cast into a single expression, covering both the (non-trivial) plastic loading and the (trivial) elastic unloading cases, yielding,

\[
\gamma = \left( \partial_{\sigma} f : C : \partial_{\sigma} f \right)^{-1} \left\langle \partial_{\sigma} f : \dot{\sigma}^{\text{trial}} \right\rangle \geq 0
\]
Continuum elastoplastic tangent tensor

For the (non-trivial) case of plastic loading, the stress rate and fourth-order continuum elastoplastic tangent tensor are given by,

\[
\dot{\sigma} = C : \dot{\varepsilon}^e = C : (\dot{\varepsilon} - \dot{\varepsilon}^p) = \dot{\sigma}^{\text{trial}} - \gamma C : \partial_\sigma f
\]

\[
= \dot{\sigma}^{\text{trial}} - (\partial_\sigma f : C : \partial_\sigma f)^{-1} (C : \partial_\sigma f) (\partial_\sigma f : \dot{\sigma}^{\text{trial}})
\]

\[
= \left( C - (\partial_\sigma f : C : \partial_\sigma f)^{-1} (C : \partial_\sigma f) \otimes (\partial_\sigma f : C) \right) : \dot{\varepsilon} := C^{ep} : \dot{\varepsilon}
\]

\[
C^{ep} := C - (\partial_\sigma f : C : \partial_\sigma f)^{-1} (C : \partial_\sigma f) \otimes (\partial_\sigma f : C)
\]
Differential constitutive equation

For *elastic loading/unloading* cases the differential constitutive equation may be written as,

\[
\text{if } f < 0 \quad \text{or} \quad f = 0 \quad \text{and} \quad \dot{f} < 0 \quad \Rightarrow \quad d\sigma = C : d\varepsilon
\]

For the *plastic loading* case the differential constitutive equation may be written as,

\[
\text{if } f = 0 \quad \text{and} \quad \dot{f} = 0 \quad \Rightarrow \quad d\sigma = C^{ep} : d\varepsilon
\]
Assignment 8.4

Consider the clamped beam of the figure. The material is elastic-perfect plastic, with elastic Young’s modulus $E$, Poisson’s coefficient $\nu = 0$ and traction/compression yield stress $\sigma_0$. Body forces are negligible. The displacements field is assumed to be,

$$u = -z \varphi'(x), \quad v = 0, \quad w = \varphi(x)$$
Assignment 8.4

1) Show that the displacements field is the solution of the elastic problem, determine the function \( \varphi(x) \) and obtain the strains and stresses.

2) Obtain the maximum elastic moment \( M_e \) and the corresponding vertical displacement \( \delta_e \) at the end of the beam.

3) Obtain the moment \( M > M_e \) as a function of the plastified zone size in an arbitrary transversal section of the beam.

4) Obtain the maximum moment (limit moment) \( M_p \) when all the points of the transversal section of the beam have plastified.

5) Plot the curve \( M \) vs \( \delta \) for \( 0 \leq \delta < \infty \).
Von Mises Yield Surface

The Von Mises yield surface may be written as,

$$f(\sigma) = \sqrt{\frac{3}{2}} \| \text{dev} \sigma \| - \sigma_Y = 0$$

The Von Mises yield surface, in terms of the stress invariants, may be written as.

$$f(\sigma) = F(J'_2) = (3J'_2)^{1/2} - \sigma_Y = 0$$
Von Mises Yield Surface

In the *space of principal stresses* the Von Mises yield surface is a *cylinder* with axis the *hydrostatic stress axis* and radius $R$ given by,

$$R := \sqrt{3} \tau_{oct} = \sqrt{2} \left( J'_2 \right)^{1/2} = \sqrt{\frac{2}{3}} \sigma_Y$$
Von Mises Yield Surface

The Von Mises yield surface has the following features:

- Does not depend on the first invariant of the stress tensor and therefore the intersection of any octahedral plane with the yield surface gives the same circumference.
- For a positive yield stress, i.e. positive radius, hydrostatic stress states will never plastify.
- Does not depend on the third invariant of the deviatoric stress tensor and therefore is an axisymmetric surface.
- Appropriate for metals, i.e. materials having the same behaviour in traction and compression.
Example 8.1

The stress state at an arbitrary section of a beam is given by,

\[
\sigma = \begin{bmatrix}
\sigma_x & \tau_{xy} & 0 \\
\tau_{xy} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Obtain the expression of the yield surface using the Von Mises model.
Example 8.1

The stress state at an arbitrary section of a beam is given by,

\[
\begin{bmatrix}
\sigma_x & \tau_{xy} & 0 \\
\tau_{xy} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

The mean stress and deviatoric stress tensor are given by,

\[
\sigma_m = \frac{1}{3} \text{tr} \sigma = \frac{1}{3} \sigma_x,
\quad [\text{dev } \sigma] =
\begin{bmatrix}
\frac{2}{3} \sigma_x & \tau_{xy} & 0 \\
\tau_{xy} & -\frac{1}{3} \sigma_x & 0 \\
0 & 0 & -\frac{1}{3} \sigma_x
\end{bmatrix}
\]
The second invariant of the deviatoric stress tensor is given by,

\[
J'_2 = \frac{1}{2} \text{dev} \sigma : \text{dev} \sigma
\]

\[
= \frac{1}{2} \left( \frac{4}{9} \sigma_x^2 + \frac{1}{9} \sigma_x^2 + \frac{1}{9} \tau_{xy}^2 + \tau_{xy}^2 \right) = \frac{1}{3} \sigma_x^2 + \tau_{xy}^2
\]

The Von Mises yield surface takes the form,

\[
f(\sigma) = (3J'_2)^{1/2} - \sigma_Y = \sqrt{\sigma_x^2 + 3\tau_{xy}^2} - \sigma_Y = 0
\]
Tresca Yield Surface

The *Tresca yield surface* may be written as,

\[ f(\sigma) = 2\tau_{\text{max}} - \sigma_Y = (\sigma_1 - \sigma_3) - \sigma_Y = 0 \]

The *Tresca yield surface*, in terms of the stress invariants, takes the form,

\[ f(\sigma) = F(J'_2, J'_3) = 0 \]
Tresca Yield Surface

In the space of principal stresses, the Tresca yield surface is an hexagonal prismatic surface with axis the hydrostatic stress axis and inscribed in the cylinder of Von Mises with radius,

\[ R := \sqrt{3\tau_{oct}} = \sqrt{2 \left( J'_2 \right)^{1/2}} = \sqrt{\frac{2}{3}} \sigma_Y \]
Von Mises and Tresca Yield Surfaces
The *Tresca yield surface* has the following features:

- Does not depend on the first invariant of the stress tensor and therefore the intersection of any octahedral plane with the yield surface gives the same hexagon.
- For a positive yield stress, hydrostatic stress states will never plastify.
- Depends on the third invariant of the deviatoric stress tensor and therefore it is *not* axisymmetric surface.
- Appropriate for metals, i.e. materials having the same behaviour in traction and compression.
Example 8.2

The stress state at an arbitrary section of a beam is given by,

\[
\begin{bmatrix}
\sigma_x & \tau_{xy} & 0 \\
\tau_{xy} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Obtain the expression of the yield surface using the Tresca model.
Example 8.2

The stress state at an arbitrary section of a beam is given by,

$$\begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The principal stresses are given by,

$$\sigma_1 = \frac{1}{2} \sigma_x + \sqrt{\frac{1}{4} \sigma_x^2 + \tau_{xy}^2}, \quad \sigma_3 = \frac{1}{2} \sigma_x - \sqrt{\frac{1}{4} \sigma_x^2 + \tau_{xy}^2}$$

The Tresca yield surface takes the form,

$$f(\sigma) = \sigma_1 - \sigma_3 = \sqrt{\sigma_x^2 + 4 \tau_{xy}^2} = \sigma_Y$$
The second invariant of the deviatoric stress tensor is given by,

\[ J'_2 = \frac{1}{2} \text{dev} \sigma : \text{dev} \sigma \]

\[ = \frac{1}{2} \left( \frac{4}{9} \sigma_x^2 + \frac{1}{9} \sigma_x^2 + \frac{1}{9} \sigma_x^2 + \tau_{xy}^2 + \tau_{xy}^2 \right) = \frac{1}{3} \sigma_x^2 + \tau_{xy}^2 \]

The Von Mises yield surface takes the form,

\[ f(\sigma) = \left( 3J'_2 \right)^{1/2} - \sigma_Y = \sqrt{\sigma_x^2 + 3\tau_{xy}^2} - \sigma_Y = 0 \]
Moh-Coulomb Yield Surface

The *Mohr-Coulomb yield surface* may be written as,

\[
f(\sigma) = \tau_n + \sigma_n \tan \phi - c = 0
\]
Moh-Coulomb Yield Surface

The normal and shear stresses at the yield surface may be written in terms of the principal stresses, yielding,

\[
\sigma_n = \frac{1}{2} (\sigma_1 + \sigma_3) + \frac{1}{2} (\sigma_1 - \sigma_3) \sin \phi, \quad \tau_n = \frac{1}{2} (\sigma_1 - \sigma_3) \cos \phi
\]
Moh-Coulomb Yield Surface

The *Mohr-Coulomb yield surface* may be written as,

\[ f(\sigma) = (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3)\sin\phi - 2c\cos\phi = 0 \]

The *Mohr-Coulomb yields surface*, in terms of the stress invariants, takes the form,

\[ F(\sigma) = F(I_1, J'_2, J'_3) = 0 \]
Moh-Coulomb Yield Surface

In the space of principal stresses, the Mohr-Coulomb yield surface is an hexagonal pyramid surface with axis the hydrostatic stress axis and vertex V on the positive side (traction) of the hydrostatic axis, at a distance $d$ of the origin given by,

$$V(c \cot \phi, c \cot \phi, c \cot \phi), \quad d = \sqrt{3}c \cot \phi$$
Moh-Coulomb Yield Surface

The Tresca yield surface may be obtained as a particular case of the Mohr-Coulomb yield surface setting,

\[ \phi = 0, \quad c = \frac{1}{2} \sigma_y \]

\[ f(\sigma) = (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3) \sin \phi - 2c \cos \phi \]

\[ = 2\sigma_m \sin \phi - 2c \cos \phi = 0 \]

\[ f(\sigma) = (\sigma_1 - \sigma_3) - 2c = (\sigma_1 - \sigma_3) - \sigma_y = 0 \]
Moh-Coulomb Yield Surface

- According to the *Mohr-Coulomb plasticity model*, positive hydrostatic stress states may plastify, but negative hydrostatic stress states will remain always elastic.
- The *Mohr-Coulomb plasticity model* may be suitable for frictional-cohesive materials, exhibiting a rather different behaviour under traction/compression stress states, such as *soils, rocks, concrete*. 
Drucker-Prager Yield Surface

The Drucker-Prager yield surface may be written as,

\[ f(\sigma) = 3\alpha \sigma_{oct} + \sqrt{\frac{3}{2}} \tau_{oct} - \beta = 0 \]

where

\[ \alpha = \frac{2 \sin \phi}{\sqrt{3 (3 - \sin \phi)}}, \quad \beta = \frac{6c \cos \phi}{\sqrt{3 (3 - \sin \phi)}} \]

The Drucker-Prager yield surface, in terms of the stress invariants, may be written as,

\[ f(\sigma) = \alpha I_1 + J_2^{1/2} - \beta = 0 \]
Drucker-Prager Yield Surface

In the space of principal stresses, the Drucker-Prager yield surface is an conical surface with axis the hydrostatic stress axis and vertex V on the positive side (traction) of the hydrostatic axis, at a distance $d$ of the origin given by,

$$V(c \cot \phi, c \cot \phi, c \cot \phi), \quad d = \sqrt{3} c \cot \phi$$
Drucker-Prager Yield Surface

The Von Mises yield surface may be obtained as a particular case of the Drucker-Prager yield surface setting,

\[ \phi = 0, \quad c = \frac{1}{2} \sigma_Y \]

\[ \alpha \bigg|_{\phi=0} = \left. \frac{2 \sin \phi}{\sqrt{3} (3 - \sin \phi)} \right|_{\phi=0} = 0, \quad \beta \bigg|_{\phi=0} = \left. \frac{6c \cos \phi}{\sqrt{3} (3 - \sin \phi)} \right|_{c=\sigma_Y/2} = \frac{\sigma_Y}{\sqrt{3}} \]

\[ f(\sigma) = \alpha I_1 + J_2'^{1/2} - \beta = J_2'^{1/2} - \frac{\sigma_Y}{\sqrt{3}} = 0 \Rightarrow \sqrt{3} J_2'^{1/2} - \sigma_Y = 0 \]
Drucker-Prager Yield Surface

- According to the Drucker-Prager plasticity model, positive hydrostatic stress states may plastify, but negative hydrostatic stress states will remain always elastic.

- The Drucker-Prager plasticity model may be suitable for frictional-cohesive materials, exhibiting a rather different behaviour under traction/compression stress states, such as soils, rocks, concrete.
**Yield Surfaces**

**Von Mises**

\[ f(\sigma) = \frac{\sqrt{3}}{2} \left| \text{dev} \sigma \right| - \sigma_Y = \left( 3J_2' \right)^{1/2} - \sigma_Y = 0 \]

**Tresca**

\[ f(\sigma) = 2\tau_{\text{max}} - \sigma_Y = (\sigma_1 - \sigma_3) - \sigma_Y = 0 \]

**Mohr-Coulomb**

\[ f(\sigma) = (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3) \sin \phi - 2c \cos \phi = 0 \]

**Drucker-Prager**

\[ f(\sigma) = 3\alpha \sigma_{\text{oct}} + \sqrt{\frac{3}{2}} \tau_{\text{oct}} - \beta = \alpha I_1 + J_2'^{1/2} - \beta = 0 \]